Contextuality: at the borders of paradox

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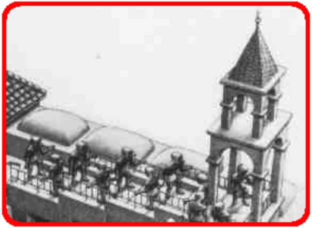
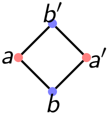
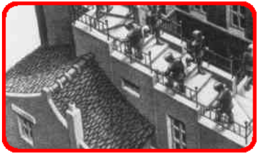
In a nutshell: where we have a family of data which is **locally consistent**, but **globally inconsistent**.

Contextuality Analogy: Local Consistency







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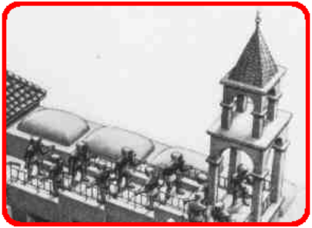

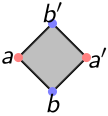

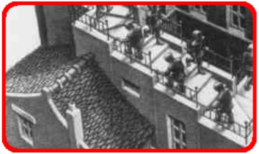

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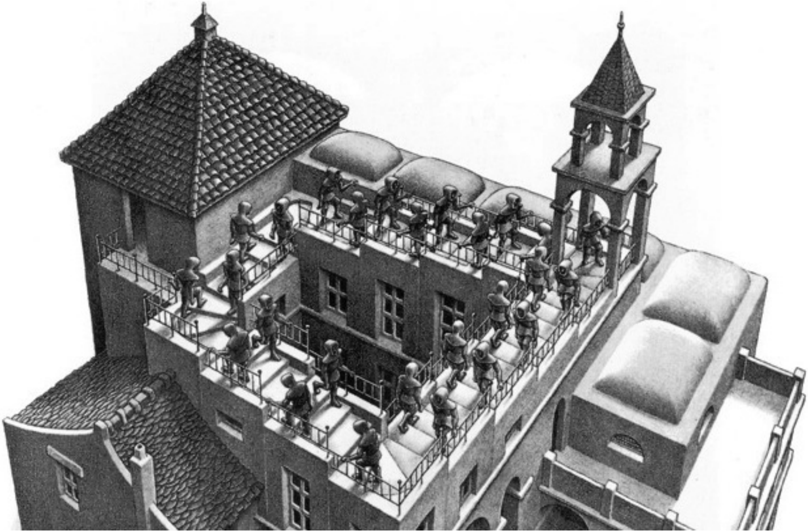

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A "transcendental deduction" of the **incompatibility** (in general) of observables.

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Our results show that it does apply, in a very direct way, to the analysis of contextuality.

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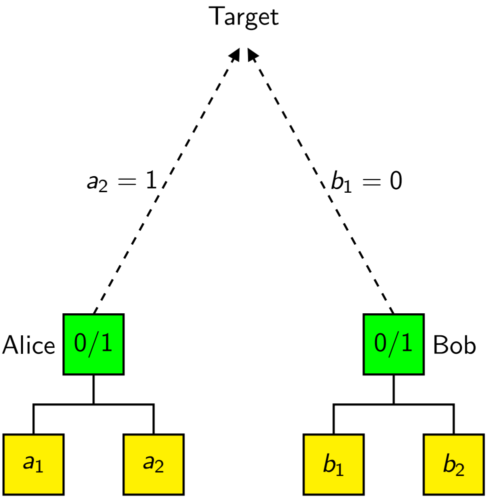
Logical and sheaf-theoretic structure also plays a key rôle in discerning a hierarchy of **degrees of contextuality**.

Alice and Bob look at bits



A Probabilistic Model Of An Experiment

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Example: The Bell Model

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a_1	b_1	$1/2$	0	0	$1/2$
a_1	b_2	$3/8$	$1/8$	$1/8$	$3/8$
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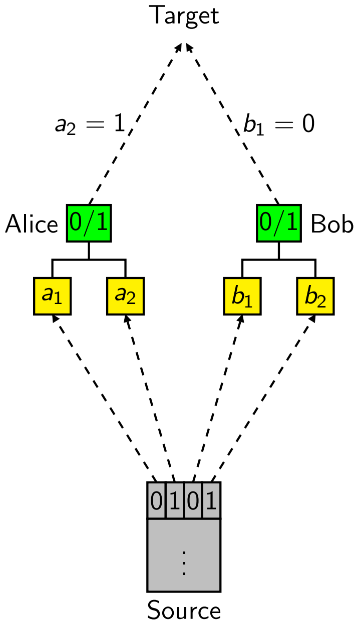
How can we explain this behaviour?

Classical Correlations: The Classical Source



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Using elementary probability theory, we can calculate:

$$p_N \;\leq\; \mathrm{Prob}(\bigvee_{i=1}^{N-1} \neg\phi_i) \;\leq\; \sum_{i=1}^{N-1} \mathrm{Prob}(\neg\phi_i) \;=\; \sum_{i=1}^{N-1}(1 - p_i) \;=\; (N-1) - \sum_{i=1}^{N-1} p_i.$$

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Hence we obtain the inequality

$$\sum_{i=1}^{N} p_i \;\leq\; N - 1.$$

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\begin{aligned}
\varphi_1 &= (a_1 \wedge b_1) \vee (\neg a_1 \wedge \neg b_1) = a_1 \leftrightarrow b_1 \\
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The violation of the logical Bell inequality is $1/4$.

Example: the Hardy model

The support of the Hardy model:

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Hence the Hardy model achieves a violation of $p_1 = \text{Prob}(a \wedge b)$ for the logical Bell inequality.

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Can we explain this behaviour using a classical source?

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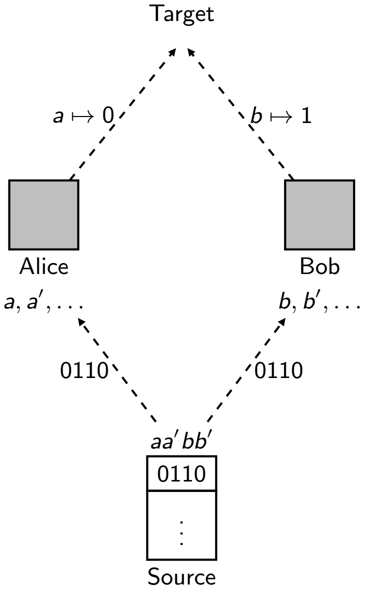
However, this view is **impossible to sustain** in the light of our **actual observations of (micro)-physical reality**.

Hidden Variables: The Mermin instruction set picture



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Hardy models: those whose support satisfies

	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
(a_1, b_1)	1			
(a_1, b_2)	0			
(a_2, b_1)	0			
(a_2, b_2)				0

Which 'instruction set' λ could the outcomes $(0,0)$ for measurements (a_1, b_1) could come? Clearly, we must have

$$\lambda : a_1 \mapsto 0, \quad b_1 \mapsto 0.$$

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Thus Hardy models are **contextual**. They cannot be explained by a classical source.

Quantum Mechanics changes the game

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Alice and Bob's choices are now of **measurement setting** (e.g. which direction to measure spin) rather than "which register to load".

A Possibilistic Model Of An Experiment

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This proves a **strong version of Bell's theorem**.

Bundle Pictures

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- Ignore precise probabilities
- Events are possible or not
- E.g. the Hardy model:

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ab	✓	✓	✓	✓
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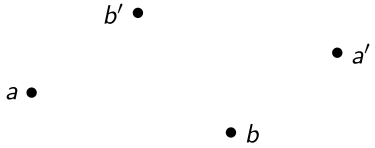

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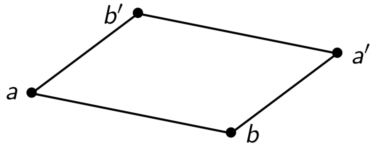

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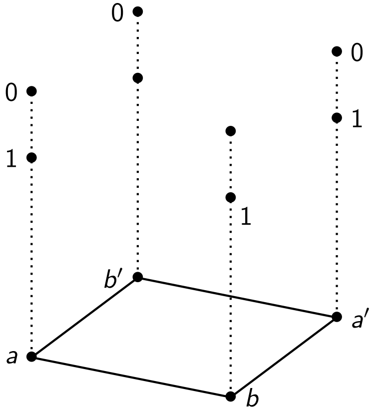

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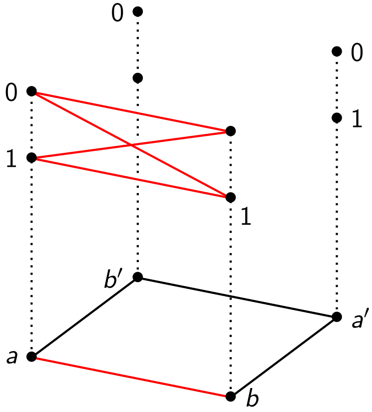

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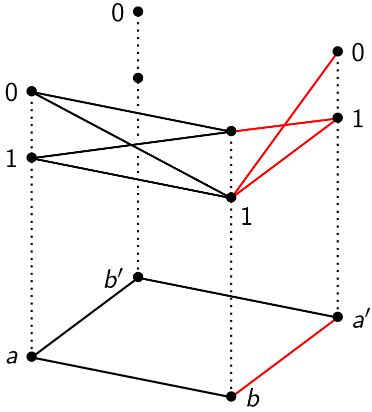

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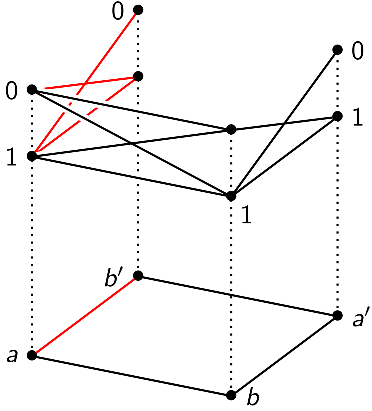

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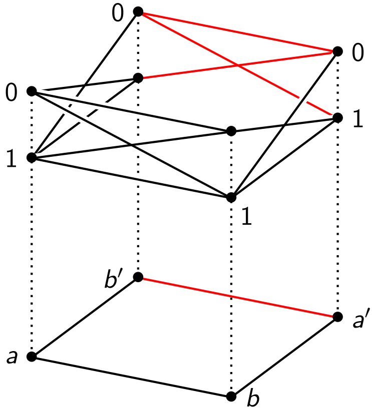

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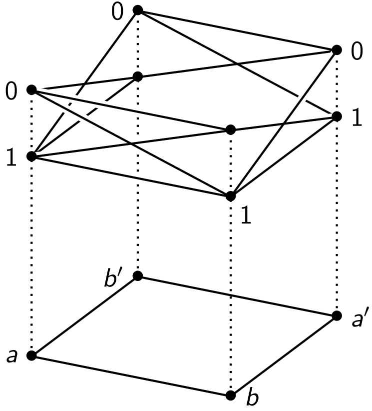

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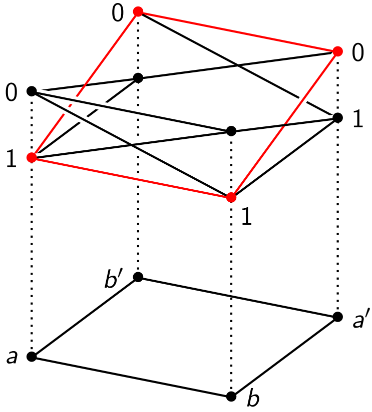

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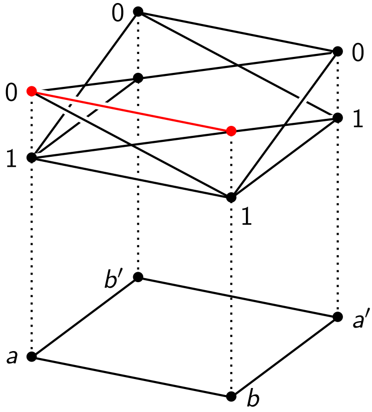

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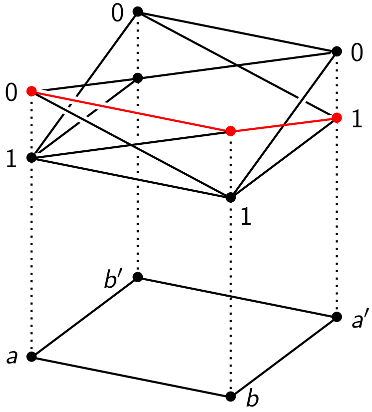

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ab	✓	✓	✓	✓
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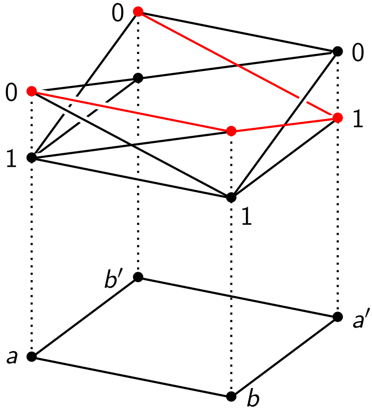

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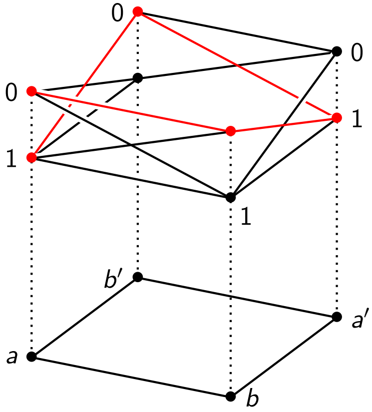

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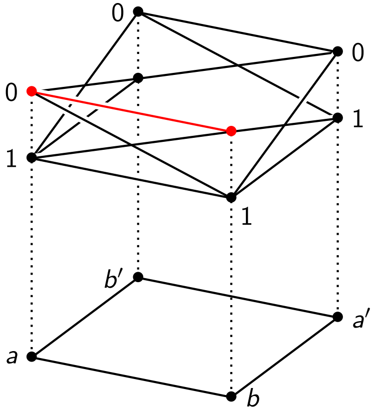
	00	01	10	11
ab	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$
ab'	$\times$	$\checkmark$	$\checkmark$	$\checkmark$
$a'b$	$\times$	$\checkmark$	$\checkmark$	$\checkmark$
$a'b'$	$\checkmark$	$\checkmark$	$\checkmark$	$\times$



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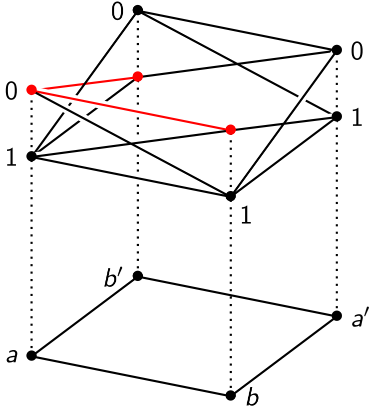
$a'b'$	✓	✓	✓	×



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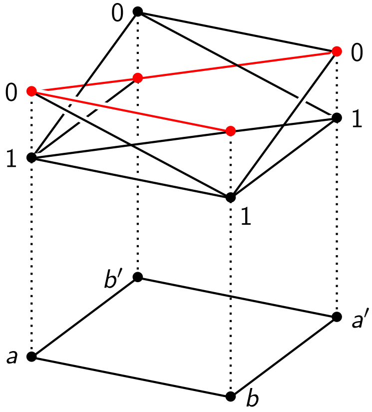

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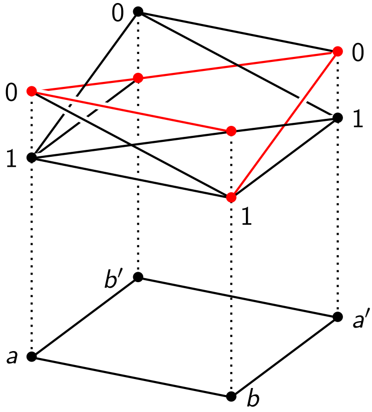


Strong Contextuality

A	B	$(0,0)$	$(1,0)$	$(0,1)$	$(1,1)$
a_1	b_1	1	0	0	1
a_1	b_2	1	0	0	1
a_2	b_1	1	0	0	1
a_2	b_2	0	1	1	0

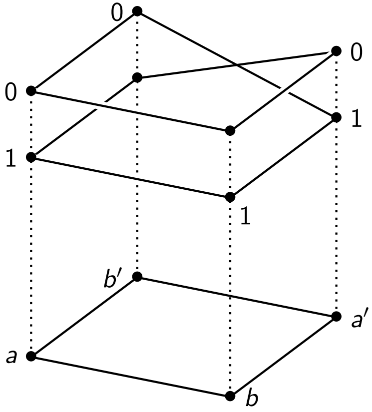
The PR Box

Bundle Pictures

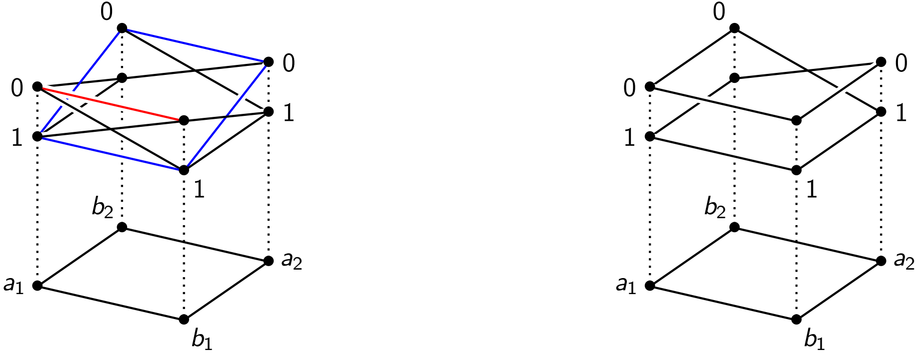
Strong Contextuality

- E.g. the PR box:

	00	01	10	11
ab	✓	×	×	✓
ab'	✓	×	×	✓
$a'b$	✓	×	×	✓
$a'b'$	×	✓	✓	×



Visualizing Contextuality



The Hardy table and the PR box as bundles

Contextuality, Logic and Paradoxes

Contextuality, Logic and Paradoxes

Liar cycles. A Liar cycle of length N is a sequence of statements

$$\begin{aligned} S_1 &: \ S_2 \text{ is true,} \\ S_2 &: \ S_3 \text{ is true,} \\ &\vdots \\ S_{N-1} &: \ S_N \text{ is true,} \\ S_N &: \ S_1 \text{ is false.} \end{aligned}$$

For $N = 1$, this is the classic Liar sentence

$$S : \ S \text{ is false.}$$

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Following Cook, Walicki et al. we can model the situation by boolean equations:

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The "paradoxical" nature of the original statements is now captured by the inconsistency of these equations.

Contextuality in the Liar; Liar cycles in the PR Box

Contextuality in the Liar; Liar cycles in the PR Box

We can regard each of these equations as fibered over the set of variables which occur in it:

$$
\begin{aligned}
\{x_1, x_2\} : \quad x_1 &= x_2 \\
\{x_2, x_3\} : \quad x_2 &= x_3 \\
&\vdots \\
\{x_{n-1}, x_n\} : \quad x_{n-1} &= x_n \\
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Up to rearrangement, **the Liar cycle of length 4 corresponds exactly to the PR box**.

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& \vdots & & \\
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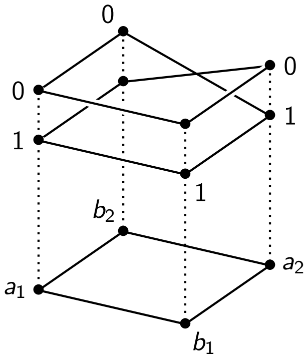
Any subset of up to $n-1$ of these equations is consistent; while the whole set is inconsistent.

Up to rearrangement, **the Liar cycle of length 4 corresponds exactly to the PR box**.

The usual reasoning to derive a contradiction from the Liar cycle corresponds precisely to the attempt to find a univocal path in the bundle diagram.

Paths to contradiction



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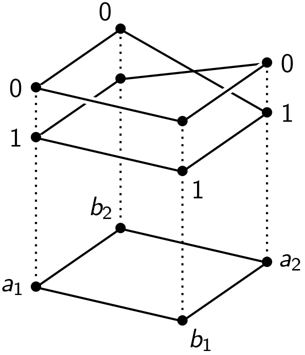

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$$a_2 = 1 \rightsquigarrow b_1 = 1 \rightsquigarrow a_1 = 1 \rightsquigarrow b_2 = 1 \rightsquigarrow a_2 = 0$$
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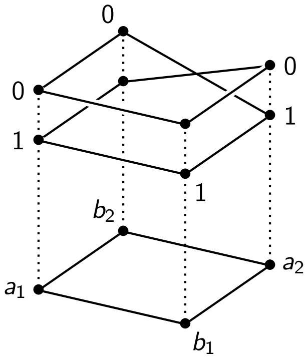

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We have discussed a specific case here, but the analysis can be generalised to a large class of examples.

The Robinson Consistency Theorem

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A classic result:

Theorem (Robinson Joint Consistency Theorem)

Let T_i be a theory over the language L_i, $i = 1, 2$. If there is no sentence ϕ in $L_1 \cap L_2$ with $T_1 \vdash \phi$ and $T_2 \vdash \neg\phi$, then $T_1 \cup T_2$ is consistent.

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This example is well-known in the quantum contextuality literature as the **Specker triangle**.

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Measurement scenarios $\langle X, \mathcal{M}, O \rangle$:

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- O is set of outcomes or values for the variables, which we take to be the same in each fibre.

We have a sheaf of sets over $\mathcal{P}(X)$, namely $\mathcal{E} :: U \longmapsto O^U$ with restriction

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A probability table can be represented by a family $\{p_C\}_{C \in \mathcal{M}}$ with p_C a probability distribution on $\mathcal{E}(C) = O^C$, where contexts C corresponds to the rows of the table.

Empirical Models

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The logical and strong forms of contextuality are concerned with **possibilities**, which can be represented by a subpresheaf $\mathcal{S}$ of $\mathcal{E}$, where for each context $U \subseteq X$, $\mathcal{S}(U) \subseteq O^U$ is the set of all possible outcomes.

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We can use this formalisation to characterize contextuality as follows.

Definition

For any empirical model $\mathcal{S}$:

- For all $C \in \mathcal{M}$ and $s \in \mathcal{S}(C)$, $\mathcal{S}$ is logically contextual at s, written $\mathsf{LC}(\mathcal{S}, s)$, if s is not a member of any compatible family.
- $\mathcal{S}$ is **strongly contextual**, written $\mathsf{SC}(\mathcal{S})$, if $\mathsf{LC}(\mathcal{S}, s)$ for all s. Equivalently, if it has no global section, *i.e.* if $\mathcal{S}(X) = \varnothing$.

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Here γ is in fact the **connecting homomorphism** of the long exact sequence.

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Proposition

The following are equivalent:

1. *The cohomology obstruction vanishes:* $\gamma(s_1) = 0$.
2. *There is a family* $\{r_i \in \mathcal{F}(C_i)\}$ *with* $s_1 = r_1$, *and for all* i, j:

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Thus non-vanishing of the obstruction provides a cohomological witness for contextuality.

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- In recent work, we obtain very general results in cases where the outcomes themselves have a module structure (over the same ring as the cohomology coefficients).
- This yields cohomological characterisations of **All-vs.-Nothing** proofs (Mermin). These account for most of the contextuality arguments in the quantum literature. In particular, we can find large classes of concrete examples in **stabiliser QM**.

Theorem

Let $\mathcal{S}$ be an empirical model on $\langle X, \mathcal{M}, R\rangle$. Then:

$$\mathsf{AvN}_R(\mathcal{S}) \Rightarrow \mathsf{SC}(\mathsf{Aff}\,\mathcal{S}) \Rightarrow \mathsf{CSC}_R(\mathcal{S}) \Rightarrow \mathsf{CSC}_{\mathbb{Z}}(\mathcal{S}) \Rightarrow \mathsf{SC}(\mathcal{S}) \; .$$

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...	...	...	...

From possibility models to databases

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Consider again the Hardy model:

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(a_1, b_1)	1	1	1	1
(a_1, b_2)	0	1	1	1
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Change of perspective:

a_1, a_2, b_1, b_2	attributes
0, 1	data values
joint outcomes of measurements	tuples

The Hardy model as a relational database

The four rows of the model turn into four **relation tables**:

a_1	b_1
0	0
0	1
1	0
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There is no **universal relation**: no table

a_1	a_2	b_1	b_2
$\vdots$	$\vdots$	$\vdots$	$\vdots$

whose projections onto $\{a_i, b_i\}$, $i = 1, 2$, yield the above four tables.

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Relational databases	measurement scenarios
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set of attributes defining a relation table	compatible set of measurements
database schema	measurement cover
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relation/set of tuples	boolean distribution on joint outcomes
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We can also consider probabilistic databases and other generalisations; cf. provenance semirings.

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For an accessible overview of Contextual Semantics, see the article in the *Logic in Computer Science* Column, Bulletin of EATCS No. 113, June 2014 (and `arXiv`).

People

Comrades in Arms in Contextual Semantics:

People

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People

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Adam Brandenburger, Lucien Hardy, Shane Mansfield, Rui Soares Barbosa, Ray Lal, Mehrnoosh Sadrzadeh, Phokion Kolaitis, Georg Gottlob, Carmen Constantin, Kohei Kishida

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- **Hardy is almost everywhere**: with bipartite exceptions, an algorithm which given an n-qubit entangled state, constructs $n + 2$ local observables leading to a logically contextual model.
- Characterization of the **face lattice** of the No-Signalling polytope as isomorphic to the support lattice.
- General characterisation of **All-versus-Nothing** arguments. Use of **sheaf cohomology** to capture contextuality for all such models. Large classes of quantum examples using stabiliser groups.

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